THE NUMERICAL SOLUTION
OF INTEGRAL-ALGEBRAIC EQUATIONS OF INDEX 1
BY POLYNOMIAL SPLINE COLLOCATION METHODS

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ABSTRACT. In this paper, we study polynomial spline collocation methods applied to a particular class of integral-algebraic equations of Volterra type. We analyse mixed systems of second and first kind integral equations. Global convergence and local superconvergence results are established.

1. INTRODUCTION

In this paper, we study numerical methods for the solution of a mixed system of Volterra integral equations of the first and second kind. More precisely, we consider the system

$$(1.1) \qquad y(t) = f(t) + \int_0^t \left(K_{11}(t,s)y(s) + K_{12}(t,s)z(s) \right) ds,$$

$$(1.2) \qquad 0 = g(t) + \int_0^t \left(K_{21}(t,s)y(s) + K_{22}(t,s)z(s) \right) ds,$$

and its nonlinear counterpart

$$(1.3) \qquad y(t) = f(t) + \int_0^t k(t,s,y(s),z(s))\, ds,$$

$$(1.4) \qquad 0 = g(t) + \int_0^t \ell(t,s,y(s),z(s))\, ds.$$

Here we assume that $t \in I = [0,T]$ and that the data functions f, g, K_{ij}, $i,j = 1,2$, k and ℓ are sufficiently smooth. Furthermore we suppose that $g(0) = 0$, $|K_{22}(t,t)| \geq k_0 > 0$ for all $t \in I$ and that the partial derivative of the function ℓ w.r. to its fourth variable $\partial_4 \ell(t,t,y,z)$ is invertible for $t \in I$ and y, z in a neighborhood of the solution. It then follows that the systems (1.1), (1.2) or (1.3), (1.4), respectively, have a continuous solution y, z on I. This can be seen easily as follows. We first differentiate equations (1.2) and (1.4) w.r. to t and consider the resulting equations as equations of the second kind for z. We formally solve for z and replace the resulting expressions in (1.1), (1.3), respectively. We thus obtain equations of the second kind for y. However it has to be pointed out that this reduction to an integral equation of the second kind for y is not practical from a numerical point of view.

Coupled systems of Volterra equations of the first and second kind arise for example in a slightly different form in problems of identification of memory kernels in heat conduction and viscoelasticity (see, e.g., [5] and [6]). As a starting point for our investigations we have chosen the form (1.1), (1.2). The systems (1.1), (1.2) and (1.3), (1.4) are also special cases of the *integral-algebraic* equations

$$(1.5) \qquad a(y) = a(y_0) + \int_0^t k(t, s, y(s)) \, ds,$$

considered in [2], where it is asssumed that $A = a_y = \partial a / \partial y$ is singular. In analogy with the theory of differential-algebraic equations (see, e.g., [4]) we call the systems (1.1), (1.2) and (1.3), (1.4) *semi-explicit index* 1 problems.

The system (1.1), (1.2) (and similarly (1.3), (1.4)) can be seen as the limit as $\varepsilon \to 0$ of the singularly perturbed problem

$$(1.6) \qquad y(t) = f(t) + \int_0^t \left(K_{11}(t, s) y(s) + K_{12}(t, s) z(s) \right) ds,$$

$$(1.7) \qquad \varepsilon z(t) = g(t) + \int_0^t \left(K_{21}(t, s) y(s) + K_{22}(t, s) z(s) \right) ds.$$

Systems of the type (1.1), (1.2) also naturally arise when computing asymptotic expansions of the solution of (1.6), (1.7) and when studying numerical methods for such problems (see [7]).

The outline of this paper follows. In Section 2 we apply polynomial spline collocation methods to (1.1), (1.2) and (1.3), (1.4). A global convergence analysis is presented in Section 3, and Section 4 deals with superconvergence results. We conclude with a numerical illustration in Section 5.

2. POLYNOMIAL SPLINE COLLOCATION

Let Π_N be a uniform partition of the interval I with grid points $t_n = nh$, $n = 0, \dots, N$, and let h be the stepsize. Define the subintervals $\sigma_0 = [t_0, t_1]$, $\sigma_n = (t_n, t_{n+1}]$, $n = 1, \dots, N - 1$. Let the collocation parameters be $0 < c_1 < c_2 < \cdots < c_m \le 1$ and the collocation points be $t_{nj} = t_n + c_j h$, $j = 1, \dots, m$, $n = 0, \dots, N - 1$. We consider polynomial spline approximations $u(t)$, $v(t)$ of the exact solution $y(t)$, $z(t)$ in the spline space

$$S_{m-1}^{(-1)}(\Pi_N) = \{ u : \ u_n = u|_{\sigma_n} \in \pi_{m-1}, \ n = 0, \dots, N - 1 \}.$$

This is the space of piecewise polynomials of degree (at most) $m - 1$. Its dimension is Nm, i.e., the same as the number of collocation points. In what follows we will consider the linear system (1.1), (1.2), but everything carries over to the nonlinear case (see Remark 4.2). We seek u and v such that the collocation equations

$$(2.1) \qquad u(t) = f(t) + \int_0^t \left(K_{11}(t, s) u(s) + K_{12}(t, s) v(s) \right) ds,$$

$$(2.2) \qquad 0 = g(t) + \int_0^t \left(K_{21}(t, s) u(s) + K_{22}(t, s) v(s) \right) ds,$$

are satisfied for $t = t_{nj}$, $j = 1, \ldots, m$, $n = 0, \ldots, N - 1$. Let $Y_{nj} = u_n(t_{nj})$ and $Z_{nj} = v_n(t_{nj})$. Since $u_n, v_n \in \pi_{m-1}$, it holds for $\tau \in (0, 1]$,

$$(2.3) \qquad u_n(t_n + \tau h) = \sum_{j=1}^{m} L_j(\tau) Y_{nj}, \qquad v_n(t_n + \tau h) = \sum_{j=1}^{m} L_j(\tau) Z_{nj}.$$

Here $L_j(\tau) = \Pi_{k \neq j}(\tau - c_k)/(c_j - c_k)$, $j = 1, \ldots, m$, denote the fundamental Lagrange polynomials. Inserting (2.3) into (2.1), (2.2), we obtain, for each $n = 0, \ldots, N - 1$, a linear system for the unknowns Y_{nj}, Z_{nj}, $j = 1, \ldots, m$,

$$(2.4) \qquad Y_{nj} = f(t_{nj}) + h \sum_{k=1}^{m} \left(\int_0^{c_j} K_{11}(t_{nj}, t_n + \tau h) L_k(\tau) \, d\tau \right) Y_{nk}$$

$$+ h \sum_{k=1}^{m} \left(\int_0^{c_j} K_{12}(t_{nj}, t_n + \tau h) L_k(\tau) \, d\tau \right) Z_{nk}$$

$$+ h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \left(\int_0^1 K_{11}(t_{nj}, t_i + \tau h) L_k(\tau) \, d\tau \right) Y_{ik}$$

$$+ h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \left(\int_0^1 K_{12}(t_{nj}, t_i + \tau h) L_k(\tau) \, d\tau \right) Z_{ik},$$

$$(2.5) \qquad 0 = g(t_{nj}) + h \sum_{k=1}^{m} \left(\int_0^{c_j} K_{21}(t_{nj}, t_n + \tau h) L_k(\tau) \, d\tau \right) Y_{nk}$$

$$+ h \sum_{k=1}^{m} \left(\int_0^{c_j} K_{22}(t_{nj}, t_n + \tau h) L_k(\tau) \, d\tau \right) Z_{nk}$$

$$+ h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \left(\int_0^1 K_{21}(t_{nj}, t_i + \tau h) L_k(\tau) \, d\tau \right) Y_{ik}$$

$$+ h \sum_{i=0}^{n-1} \sum_{k=1}^{m} \left(\int_0^1 K_{22}(t_{nj}, t_i + \tau h) L_k(\tau) \, d\tau \right) Z_{ik}.$$

For the nonlinear system (1.3), (1.4) we obtain of course a nonlinear system for Y_{nj}, Z_{nj}, $j = 1, \ldots, m$. In the gridpoints, the numerical solutions are defined as

$$(2.6) \qquad y_0 = f(t_0), \qquad z_0 = \sum_{j=1}^{m} L_j(0) Z_{0j},$$

$$(2.7) \qquad y_{n+1} = u_n(t_{n+1}), \qquad z_{n+1} = v_n(t_{n+1}), \qquad n = 0, \ldots, N - 1.$$

For the numerical solution of these systems, the integrals in (2.4), (2.5) have to be approximated by appropriate quadrature rules. Here we will use

$$(2.8) \qquad \int_0^{c_i} f(\tau) \, d\tau \approx \sum_{j=1}^{m} a_{ij} f(c_j), \qquad i = 1, \ldots, m,$$

$$(2.9) \qquad \int_0^1 f(\tau) \, d\tau \approx \sum_{j=1}^{m} b_j f(c_j),$$

where the coefficients a_{ij} and b_j are defined by $a_{ij} = \int_0^{c_i} L_j(\tau) \, d\tau$, $b_j = \int_0^1 L_j(\tau) \, d\tau$, $i, j = 1, \ldots, m$. The resulting approximations of y, z are denoted by $\widehat{u}$, $\widehat{v}$ and are

defined by

$$
(2.10) \quad \widehat{Y}_{nj} = f(t_{nj}) + h \sum_{k=1}^{m} a_{jk} K_{11}(t_{nj}, t_{nk}) \widehat{Y}_{nk} + h \sum_{k=1}^{m} a_{jk} K_{12}(t_{nj}, t_{nk}) \widehat{Z}_{nk}
$$

$$
+ h \sum_{i=0}^{n-1} \sum_{k=1}^{m} b_k K_{11}(t_{nj}, t_{ik}) \widehat{Y}_{ik} + h \sum_{i=0}^{n-1} \sum_{k=1}^{m} b_k K_{12}(t_{nj}, t_{ik}) \widehat{Z}_{ik},
$$

$$
(2.11) \quad 0 = g(t_{nj}) + h \sum_{k=1}^{m} a_{jk} K_{21}(t_{nj}, t_{nk}) \widehat{Y}_{nk} + h \sum_{k=1}^{m} a_{jk} K_{22}(t_{nj}, t_{nk}) \widehat{Z}_{nk}
$$

$$
+ h \sum_{i=0}^{n-1} \sum_{k=1}^{m} b_k K_{21}(t_{nj}, t_{ik}) \widehat{Y}_{ik} + h \sum_{i=0}^{n-1} \sum_{k=1}^{m} b_k K_{22}(t_{nj}, t_{ik}) \widehat{Z}_{ik},
$$

and equations similar to (2.3), (2.6) and (2.7). The internal stages $\widehat{Y}_{nj}$ and $\widehat{Z}_{nj}$ of this discretized collocation method are thus equivalent to those of an extended implicit Volterra-Runge-Kutta method (see [1]).

3. GLOBAL CONVERGENCE ANALYSIS

Let $\| \cdot \|_\infty$ be the supremum norm. We have the following global convergence result:

Theorem 3.1. *Consider the polynomial spline approximations u, v in $S_{m-1}^{(-1)}(\Pi_N)$ to the solution y, z of the system (1.1), (1.2) and defined by (2.3)–(2.5). For every choice of c_m, $(0 < c_m \leq 1)$, the collocation approximation u converges to the solution y. If $c_m = 1$, the collocation approximation v converges to the solution z, and if $c_m < 1$, the collocation approximation v converges to the solution z for any $m \geq 1$ if and only if*

$$
(3.1) \qquad -1 \leq R(\infty) = (-1)^m \prod_{i=1}^{m} \frac{1 - c_i}{c_i} \leq 1.
$$

Moreover, the following error estimates hold:

$$
\| y - u \|_\infty = O(h^m),
$$

$$
\| z - v \|_\infty = \begin{cases} O(h^m) & \text{if } c_m = 1, \\ O(h^m) & \text{if } c_m < 1 \text{ and } -1 \leq R(\infty) < 1, \\ O(h^{m-1}) & \text{if } c_m < 1 \text{ and } R(\infty) = 1, \end{cases}
$$

as $h \to 0$ with $Nh \leq const$.

Proof. It holds for the exact solution that

$$
y(t_n + \tau h) = \sum_{j=1}^{m} L_j(\tau) y(t_{nj}) + r_n(\tau), \qquad r_n(\tau) = h^m \frac{y^{(m)}(\eta_n(\tau))}{m!} \prod_{i=1}^{m} (\tau - c_i),
$$

$$
z(t_n + \tau h) = \sum_{j=1}^{m} L_j(\tau) z(t_{nj}) + s_n(\tau), \qquad s_n(\tau) = h^m \frac{z^{(m)}(\zeta_n(\tau))}{m!} \prod_{i=1}^{m} (\tau - c_i),
$$

with $\tau \in [0, 1]$. It follows that the errors $e = y - u$ and $\varepsilon = z - v$ have the following representation

$$(3.2) \qquad e_n(t_n + \tau h) = \sum_{j=1}^{m} L_j(\tau) e_n(t_{nj}) + r_n(\tau),$$

$$(3.3) \qquad \varepsilon_n(t_n + \tau h) = \sum_{j=1}^{m} \dot{L}_j(\tau) \varepsilon_n(t_{nj}) + s_n(\tau),$$

where $e_n = e|_{\sigma_n}$ and $\varepsilon_n = \varepsilon|_{\sigma_n}$. On the other hand, the errors satisfy the system

$$(3.4) \qquad e_n(t_{nj}) = \int_0^{t_{nj}} (K_{11}(t_{nj}, s)e(s) + K_{12}(t_{nj}, s)\varepsilon(s)) \, ds,$$

$$(3.5) \qquad 0 = \int_0^{t_{nj}} (K_{21}(t_{nj}, s)e(s) + K_{22}(t_{nj}, s)\varepsilon(s)) \, ds.$$

This gives

(3.6)

$$e_n(t_{nj}) = h \int_0^{c_j} (K_{11}(t_{nj}, t_n + \tau h)e_n(t_n + \tau h) + K_{12}(t_{nj}, t_n + \tau h)\varepsilon_n(t_n + \tau h)) \, d\tau$$

$$+ h \sum_{i=0}^{n-1} \int_0^1 (K_{11}(t_{nj}, t_i + \tau h)e_i(t_i + \tau h) + K_{12}(t_{nj}, t_i + \tau h)\varepsilon_i(t_i + \tau h)) \, d\tau,$$

(3.7)

$$0 = h \int_0^{c_j} (K_{21}(t_{nj}, t_n + \tau h)e_n(t_n + \tau h) + K_{22}(t_{nj}, t_n + \tau h)\varepsilon_n(t_n + \tau h)) \, d\tau$$

$$+ h \sum_{i=0}^{n-1} \int_0^1 (K_{21}(t_{nj}, t_i + \tau h)e_i(t_i + \tau h) + K_{22}(t_{nj}, t_i + \tau h)\varepsilon_i(t_i + \tau h)) \, d\tau.$$

We now rewrite (3.7) with n replaced by $n - 1$ and $j = m$, subtract this equation from (3.7) and divide by h. (This process is the numerical analog of transforming a first kind equation into a second kind equation by differentation.) We obtain

(3.8)

$$\int_0^{c_j} (K_{21}(t_{nj}, t_n + \tau h)e_n(t_n + \tau h) + K_{22}(t_{nj}, t_n + \tau h)\varepsilon_n(t_n + \tau h)) \, d\tau$$

$$= \int_0^{c_m} (K_{21}(t_{n-1m}, t_{n-1} + \tau h)e_{n-1}(t_{n-1} + \tau h)$$

$$+ K_{22}(t_{n-1m}, t_{n-1} + \tau h)\varepsilon_{n-1}(t_{n-1} + \tau h)) \, d\tau$$

$$- \sum_{i=0}^{n-1} \int_0^1 (K_{21}(t_{nj}, t_i + \tau h)e_i(t_i + \tau h) + K_{22}(t_{nj}, t_i + \tau h)\varepsilon_i(t_i + \tau h)) \, d\tau$$

$$+ \sum_{i=0}^{n-2} \int_0^1 (K_{21}(t_{n-1m}, t_i + \tau h)e_i(t_i + \tau h)$$

$$+ K_{22}(t_{n-1m}, t_i + \tau h)\varepsilon_i(t_i + \tau h)) \, d\tau.$$

We now have to distinguish between two cases, $c_m = 1$ and $c_m < 1$.

Case I: $c_m = 1$. Now (3.8) becomes

(3.9)
$$\int_0^{c_j} (K_{21}(t_{nj}, t_n + \tau h)e_n(t_n + \tau h) + K_{22}(t_{nj}, t_n + \tau h)\varepsilon_n(t_n + \tau h)) \, d\tau$$

$$= \sum_{i=0}^{n-1} \int_0^1 (K_{21}(t_{n-1m}, t_i + \tau h)e_i(t_i + \tau h) + K_{22}(t_{n-1m}, t_i + \tau h)\varepsilon_i(t_i + \tau h)) \, d\tau$$

$$- \sum_{i=0}^{n-1} \int_0^1 (K_{21}(t_{nj}, t_i + \tau h)e_i(t_i + \tau h) + K_{22}(t_{nj}, t_i + \tau h)\varepsilon_i(t_i + \tau h)) \, d\tau.$$

Using

(3.10)
$$K_{k\ell}(t_{n-1m}, t_i + \tau h) - K_{k\ell}(t_{nj}, t_i + \tau h) = h \, \partial_1 K_{kl}(\cdot, t_i + \tau h), \qquad k, \ell = 1, 2,$$

with $\cdot$ between t_{n-1m} and t_{nj},

(3.11)
$$\int_0^{c_j} K_{22}(t_{nj}, t_n + \tau h)L_k(\tau) \, d\tau = K_{22}(t_n, t_n) \, a_{jk} + O(h), \qquad j, k = 1, \dots, m,$$

and with (3.2) and (3.3), we arrive at

$$\sum_{k=1}^m \left(\int_0^{c_j} K_{21}(t_{nj}, t_n + \tau h)L_k(\tau) \, d\tau \right) e_n(t_{nk})$$

$$+ \sum_{k=1}^m (K_{22}(t_n, t_n)a_{jk} + O(h)) \, \varepsilon_n(t_{nk})$$

$$= h \sum_{i=0}^{n-1} \sum_{k=1}^m \left(\int_0^1 \partial_1 K_{21}(\cdot, t_i + \tau h)L_k(\tau) \, d\tau \right) e_i(t_{ik})$$

$$+ h \sum_{i=0}^{n-1} \sum_{k=1}^m \left(\int_0^1 \partial_1 K_{22}(\cdot, t_i + \tau h)L_k(\tau) \, d\tau \right) \varepsilon_i(t_{ik}) + O(h^m)$$

We also rewrite (3.6) using (3.2) and (3.3)

$$(3.12) \quad e_n(t_{nj}) = h \sum_{k=1}^m \left(\int_0^{c_j} K_{11}(t_{nj}, t_n + \tau h)L_k(\tau) \, d\tau \right) e_n(t_{nk})$$

$$+ h \sum_{k=1}^m \left(\int_0^{c_j} K_{12}(t_{nj}, t_n + \tau h)L_k(\tau) \, d\tau \right) \varepsilon_n(t_{nk})$$

$$+ h \sum_{i=0}^{n-1} \sum_{k=1}^m \left(\int_0^1 K_{11}(t_{nj}, t_i + \tau h)L_k(\tau) \, d\tau \right) e_i(t_{ik})$$

$$+ h \sum_{i=0}^{n-1} \sum_{k=1}^m \left(\int_0^1 K_{12}(t_{nj}, t_i + \tau h)L_k(\tau) \, d\tau \right) \varepsilon_i(t_{ik}) + O(h^m).$$

Let $E_n = (e_n(t_{n1}), \dots, e_n(t_{nm}))^\top$ and $\epsilon_n = (\varepsilon_n(t_{n1}), \dots, \varepsilon_n(t_{nm}))^\top$, $n = 0, \dots,$ $N-1$. Then the previous two equations can be rewritten in matrix notation

$$\begin{pmatrix} I - hK_{11}^{(n,n)} & -hK_{12}^{(n,n)} \\ K_{21}^{(n,n)} & K_{22}(t_n, t_n)\, A + O(h) \end{pmatrix} \begin{pmatrix} E_n \\ \epsilon_n \end{pmatrix}$$
$$= h \sum_{i=0}^{n-1} \begin{pmatrix} K_{11}^{(n,i)} & K_{12}^{(n,i)} \\ \widetilde{K}_{21}^{(n,i)} & \widetilde{K}_{22}^{(n,i)} \end{pmatrix} \begin{pmatrix} E_i \\ \epsilon_i \end{pmatrix} + O(h^m),$$

where the meaning of the matrices $K_{k\ell}^{(n,i)}$ and $\widetilde{K}_{k\ell}^{(n,i)}$ is clear. Since $|K_{22}(t,t)| \geq k_0 > 0$ for all $t \in I$, the inverse of the matrix on the left hand side exists and is bounded if h is sufficiently small. It then follows from Gronwall's inequality that

$$\|E_n\| = \max_{j=1,\dots,m} \{|e_n(t_{nj})|\} = O(h^m), \qquad \|\epsilon_n\| = \max_{j=1,\dots,m} \{|\varepsilon_n(t_{nj})|\} = O(h^m),$$

and the result follows from (3.2) and (3.3).

Case II: $c_m < 1$. Using (3.8) and (3.10) we arrive at

$$\int_0^{c_j} \left(K_{21}(t_{nj}, t_n + \tau h) e_n(t_n + \tau h) + K_{22}(t_{nj}, t_n + \tau h)\varepsilon_n(t_n + \tau h) \right) d\tau$$

$$= \int_0^{c_m} \left(K_{21}(t_{n-1 m}, t_{n-1} + \tau h) e_{n-1}(t_{n-1} + \tau h) \right.$$
$$\left. + K_{22}(t_{n-1 m}, t_{n-1} + \tau h)\varepsilon_{n-1}(t_{n-1} + \tau h) \right) d\tau$$

$$- \int_0^1 \left(K_{21}(t_{nj}, t_{n-1} + \tau h) e_{n-1}(t_{n-1} + \tau h) \right.$$
$$\left. + K_{22}(t_{nj}, t_{n-1} + \tau h)\varepsilon_{n-1}(t_{n-1} + \tau h) \right) d\tau$$

$$+ h \sum_{i=0}^{n-2} \int_0^1 \left(\partial_1 K_{21}(\cdot, t_i + \tau h) e_i(t_i + \tau h) + \partial_1 K_{22}(\cdot, t_i + \tau h)\varepsilon_i(t_i + \tau h) \right) d\tau.$$

We now proceed as in the previous case. We insert (3.2) and (3.3), use (3.11),

(3.13)
$$\int_0^1 K_{22}(t_{nj}, t_{n-1} + \tau h) L_k(\tau)\, d\tau = K_{22}(t_{n-1}, t_{n-1})\, b_k + O(h), \qquad j, k = 1, \dots, m,$$

and similar expressions for K_{21}, divide by $K_{22}(t_n, t_n)$ and use

$$\frac{K_{22}(t_{n-1}, t_{n-1})}{K_{22}(t_n, t_n)} = 1 + O(h).$$

We thus obtain

$$\sum_{k=1}^{m} \left(\int_0^{c_j} \frac{K_{21}(t_{nj}, t_n + \tau h)}{K_{22}(t_n, t_n)} L_k(\tau)\, d\tau \right) e_n(t_{nk})$$

$$+ \sum_{k=1}^{m} \left(a_{jk} + O(h) \right) \varepsilon_n(t_{nk})$$

$$= \sum_{k=1}^{m} \left(\frac{K_{21}(t_{n-1}, t_{n-1})}{K_{22}(t_n, t_n)} a_{mk} + O(h) \right) e_{n-1}(t_{n-1\,k})$$

$$+ \sum_{k=1}^{m} \left(a_{mk} + O(h) \right) \varepsilon_{n-1}(t_{n-1\,k})$$

$$- \sum_{k=1}^{m} \left(\frac{K_{21}(t_{n-1}, t_{n-1})}{K_{22}(t_n, t_n)} b_k + O(h) \right) e_{n-1}(t_{n-1\,k})$$

$$- \sum_{k=1}^{m} \left(b_k + O(h) \right) \varepsilon_{n-1}(t_{n-1\,k})$$

$$+ h \sum_{i=0}^{n-2} \sum_{k=1}^{m} \left(\int_0^1 \frac{\partial_1 K_{21}(\cdot, t_i + \tau h)}{K_{22}(t_n, t_n)} L_k(\tau)\, d\tau \right) e_i(t_{ik})$$

$$+ h \sum_{i=0}^{n-2} \sum_{k=1}^{m} \left(\int_0^1 \frac{\partial_1 K_{22}(\cdot, t_i + \tau h)}{K_{22}(t_n, t_n)} L_k(\tau)\, d\tau \right) \varepsilon_i(t_{ik}) + O(h^m).$$

Together with (3.12) this gives in matrix notation

(3.14)
$$\begin{pmatrix} I - h K_{11}^{(n,n)} & -h K_{12}^{(n,n)} \\ \widehat{K}_{21}^{(n,n)} & A + O(h) \end{pmatrix} \begin{pmatrix} E_n \\ \epsilon_n \end{pmatrix}$$

$$= \begin{pmatrix} h K_{11}^{(n,n-1)} & h K_{12}^{(n,n-1)} \\ \frac{K_{21}(t_{n-1}, t_{n-1})}{K_{22}(t_n, t_n)} M_1 + O(h) & M_1 + O(h) \end{pmatrix} \begin{pmatrix} E_{n-1} \\ \epsilon_{n-1} \end{pmatrix}$$

$$+ h \sum_{i=0}^{n-2} \begin{pmatrix} K_{11}^{(n,i)} & K_{12}^{(n,i)} \\ \overline{K}_{21}^{(n,i)} & \overline{K}_{22}^{(n,i)} \end{pmatrix} \begin{pmatrix} E_i \\ \epsilon_i \end{pmatrix} + O(h^m),$$

where $M_1 = \mathbb{1}_m e_m^\top A - \mathbb{1}_m b^\top$ and $\mathbb{1}_m = (1, \dots, 1)^\top$, $e_m = (0, \dots, 0, 1)^\top \in \mathbb{R}^m$. It holds that the inverse of the matrix on the left hand side has the form

$$\begin{pmatrix} I - h K_{11}^{(n,n)} & -h K_{12}^{(n,n)} \\ \widehat{K}_{21}^{(n,n)} & A + O(h) \end{pmatrix}^{-1} = \begin{pmatrix} I + O(h) & O(h) \\ \widehat{\widehat{K}}_{21}^{(n,n)} & A^{-1} + O(h) \end{pmatrix}$$

if h is sufficiently small and we have

$$\begin{pmatrix} I - h K_{11}^{(n,n)} & -h K_{12}^{(n,n)} \\ \widehat{K}_{21}^{(n,n)} & A + O(h) \end{pmatrix}^{-1} \begin{pmatrix} h K_{11}^{(n,n-1)} & h K_{12}^{(n,n-1)} \\ \frac{K_{21}(t_{n-1}, t_{n-1})}{K_{22}(t_n, t_n)} M_1 + O(h) & M_1 + O(h) \end{pmatrix}$$

$$= \begin{pmatrix} O(h) & O(h) \\ \frac{K_{21}(t_{n-1}, t_{n-1})}{K_{22}(t_n, t_n)} M_0 + O(h) & M_0 + O(h) \end{pmatrix},$$

where $M_0 = A^{-1}M_1 = A^{-1}\mathbb{1}_m(e_m^\top A - b^\top)$. Thus (3.14) becomes

$$\begin{pmatrix} E_n \\ \epsilon_n \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{K_{21}(t_{n-1},t_{n-1})}{K_{22}(t_n,t_n)}M_0 & M_0 \end{pmatrix} \begin{pmatrix} E_{n-1} \\ \epsilon_{n-1} \end{pmatrix} + h \sum_{i=0}^{n-1} \begin{pmatrix} M_2 & M_3 \\ M_4 & M_5 \end{pmatrix} \begin{pmatrix} E_i \\ \epsilon_i \end{pmatrix} + O(h^m),$$

where M_i, $i = 2, \ldots, 5$, denote bounded matrices. Since the matrix multiplying $(E_{n-1}, \epsilon_{n-1})^\top$ is diagonalizable, we can conclude as in [9] with the help of the following lemma and Lemma 6 of [3]. □

Lemma 3.1. *Let M_0 be defined as above. Then M_0 has rank one and its only nonzero eigenvalue is*

$$R(\infty) = (-1)^m \prod_{i=1}^m \frac{1 - c_i}{c_i} ,$$

where $R(z) = 1 + zb^\top(I - zA)^{-1}\mathbb{1}_m$ denotes the stability function of the Runge-Kutta method (c, A, b).

Proof of Lemma 3.1. M_0 has rank one because $\mathbb{1}_m(e_m^\top A - b^\top)$ has rank one. Therefore the only nonzero eigenvalue of M_0 is equal to its trace. Let $A^{-1} = (\omega_{ij})_{i,j=1}^m$. Then

$$\text{tr}(M_0) = \sum_{i=1}^m (a_{mi} - b_i) \sum_{j=1}^m \omega_{ij} = 1 - b^\top A^{-1} \mathbb{1}_m = R(\infty).$$

The result of the lemma now follows from Theorem IV.3.10 of [4]. □

Remark 3.1. Using standard techniques it can be shown that the results of Theorem 3.1 also hold for the discretized collocation approximations $\widehat{u}$ and $\widehat{v}$.

4. SUPERCONVERGENCE RESULTS

In this section we will show that for an adequate choice of the collocation parameters c_j, the order of convergence for u is higher than m in the gridpoints t_n (superconvergence). Since for Volterra equations of the first kind superconvergence does in general not occur (see [1]), we cannot expect a higher order of convergence for v in the gridpoints.

Let the residuals δ_i, $i = 1, 2$, be defined by

$$(4.1) \qquad u(t) = f(t) - \delta_1(t) + \int_0^t (K_{11}(t,s)u(s) + K_{12}(t,s)v(s))\,ds,$$

$$(4.2) \qquad 0 = g(t) - \delta_2(t) + \int_0^t (K_{21}(t,s)u(s) + K_{22}(t,s)v(s))\,ds,$$

for all $t \in I$. It follows from (2.1), (2.2) that $\delta_i(t) = 0$, $i = 1, 2$, if $t = t_{nj}$, i.e., the residuals vanish in the collocation points. Moreover the residuals δ_i are smooth on each subinterval σ_n and δ_2 is continuous on I.

The collocation errors now verify for $t \in I$,

$$(4.3) \qquad e(t) = \delta_1(t) + \int_0^t (K_{11}(t,s)e(s) + K_{12}(t,s)\varepsilon(s))\,ds,$$

$$(4.4) \qquad 0 = \delta_2(t) + \int_0^t (K_{21}(t,s)e(s) + K_{22}(t,s)\varepsilon(s))\,ds.$$

Our aim is to write e in terms of the residuals δ_1 and δ_2. Differentiating (4.4) and using resolvent representations for ε and e, one easily shows that there exist sufficiently smooth functions $L(t,s)$ and $M(t,s)$ such that

$$(4.5) \qquad e(t) = \delta_1(t) + \int_0^t L(t,s)\delta_1(s)\,ds + \int_0^t M(t,s)\delta_2'(s)\,ds, \qquad t \in I.$$

Here δ_2' denotes the right derivative of δ_2. Hence for $t = t_n$, we have with integration by parts

$$(4.6) \qquad e(t_n) = \delta_1(t_n) + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left(L(t_n,s)\delta_1(s) + M(t_n,s)\delta_2'(s) \right) ds$$

$$= \delta_1(t_n) + \sum_{i=0}^{n-1} \Bigg(M(t_n, t_{i+1})\delta_2(t_{i+1}) - M(t_n, t_i)\delta_2(t_i)$$

$$(4.7) \qquad\qquad\qquad + \int_{t_i}^{t_{i+1}} \left(L(t_n,s)\delta_1(s) - \partial_2 M(t_n,s)\delta_2(s) \right) ds \Bigg).$$

We proceed as for Volterra equations of the second kind (see [1]). The integrals in (4.7) are approximated by quadrature formulas based on (2.9). Since the residuals vanish in the collocation points, these integrals are equal to the quadrature errors $E_{ni}^{(1)}$ and $E_{ni}^{(2)}$. It holds that

$$(4.8) \qquad e(t_n) = \delta_1(t_n) + M(t_n, t_n)\delta_2(t_n) + h \sum_{i=0}^{n-1} (E_{ni}^{(1)} + E_{ni}^{(2)}).$$

If $c_m = 1$, we have $\delta_i(t_n) = 0$, $i = 1, 2$, and the error $e(t_n)$ is equal to the quadrature error induced by the formula (2.9). Therefore superconvergence can be obtained for a suitable choice of the collocation parameters c_j. For $c_m < 1$, we have in general that $\delta_i(t_n) \neq 0$, $i = 1, 2$, (compare [1]) and we do not have superconvergence. We have thus showed the following theorem.

Theorem 4.1. *If the collocation parameters c_j are the Radau II points for $(0, 1]$ (i.e., the zeros of $P_{m-1}(2s - 1) - P_m(2s - 1)$ where P_m denotes the Legendre polynomial of degree m), then*

$$\max\{|y(t_n) - u(t_n)| : \ n = 0, \dots, N\} = O(h^{2m-1}),$$

as $h \to 0$ with $Nh \leq const.$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Remark 4.1. It is known that superconvergence does not occur for the collocation approximation to the solution of a Volterra equation of the second kind if the c_j are the Gauss points (cf. Theorem 5.3.3(c) of [1]). However the full order of superconvergence can be recovered for the iterated collocation approximation. For the system of integral-algebraic equations (1.1), (1.2) considered in this paper, we do not obtain the full order of superconvergence for the iterated collocation approximation. More precisely let u, $v \in S_{m-1}^{(-1)}(Z_N)$ be the polynomial spline approximations to y, z as defined in Section 2. We define the *iterated* collocation approximation u^I by

$$(4.9) \qquad u^I(t) = f(t) + \int_0^t \left(K_{11}(t,s)u(s) + K_{22}(t,s)v(s) \right) ds, \qquad t \in I.$$

Together with (4.1) it follows that

$$(4.10) \qquad\qquad u^I(t) - u(t) = \delta_1(t), \qquad t \in I,$$

i.e., the iterated collocation approximation u^I and the collocation approximation u coincide in the collocation points t_{nj}. Iterated collocation is therefore only interesting for methods with $c_m < 1$. Let $e^I = y - u^I$ be the iterated error. In view of (4.10) and (4.5) we have that

$$e^I(t) = e(t) - \delta_1(t) = \int_0^t L(t,s)\delta_1(s)\,ds + \int_0^t M(t,s)\delta_2'(s)\,ds, \qquad t \in I.$$

We continue as we did before for e. However due to the presence of $\delta_2(t_n)$ (cf. (4.8)), we cannot expect high order superconvergence in the case where $c_m < 1$. Nevertheless it is interesting to note that numerical evidence shows that for Gauss methods $|y(t_n) - u^I(t_n)|$ is $O(h^{m+1})$ if m is odd and $O(h^m)$ if m is even.

Remark 4.2. Using standard techniques, one shows that all previous convergence results also hold for the nonlinear system (1.3), (1.4).

Remark 4.3. In case of a pure equation of the second kind, (i.e., (1.1) without the z-component and without (1.2)), the result of Theorem 3.1 reduces to the result of Theorem 5.3.2 of [1]: one observes global convergence of order m for y. Theorem 4.1 reduces to the result of Theorem 5.3.3(a) of [1]: for Radau II collocation parameters one has superconvergence of order $2m - 1$ for y at the gridpoints. Moreover one has local superconvergence for the iterated collocation approximation to y (Radau II and Gauss parameters).

In case of a pure equation of the first kind (i.e., (1.2) without the y-component and without (1.1)), the result of Theorem 3.1 can be compared to the result of Theorem 5.5.1a of [1]. Moreover, no local superconvergence for z can be achieved (cf. Theorem 5.5.2(b) of [1]).

The convergence estimates of Theorems 3.1 and 4.1 are similar to those for the convergence of Runge-Kutta methods for semi-explicit differential-algebraic equations of index 1: except for stiffly accurate methods, one generally observes an order reduction for the z-component (for details see, e.g., Theorem VI.1.1 of [4]).

5. NUMERICAL ILLUSTRATION

We solved (1.3), (1.4) on $[0,1]$ with $k(t,s,y,z) = \exp(t-s)y^2z$, $\ell(t,s,y,z) = (1+t-s)yz$ and f, g such that the exact solution is $y(t) = \exp(-t)$, $z(t) = \cos(t)$. We collocated at the Radau II points with $m = 1,\dots,4$ and $N = 4,8,16,32,64$. We computed the observed orders of convergence from the maximum errors e and ε at the gridpoints. The results are listed in Tables 1 and 2 and they nicely confirm Theorems 3.1 and 4.1.

TABLE 1. Orders of convergence for u

N	8	16	32	64
$m = 1$	1.13	1.06	1.03	1.01
$m = 2$	2.99	3.00	3.00	3.00
$m = 3$	4.96	4.99	5.00	5.00
$m = 4$	6.99	7.00	7.00	7.00

TABLE 2. Orders of convergence for v

N	8	16	32	64
$m = 1$	1.94	1.31	1.13	1.06
$m = 2$	2.10	2.04	2.02	2.01
$m = 3$	3.11	3.05	3.03	3.01
$m = 4$	4.11	4.00	3.98	3.99

Remark 5.1. A comparison of spline collocation and pseudospectral methods for integral-algebraic equations (1.3), (1.4) can be found in [8].

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